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Event-Symmetry for Superstrings

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I apply the principle of event-symmetry to simple string models and discuss how these lead to the conviction that multiple quantization is linked to dimension. It may be that string theory has to be formulated in the absence of space-time, which will then emerge as a derived property of the dynamics. Another interpretation of the event-symmetric approach which embodies this is that instantons are fundamental. Just as solitons may be dual to fundamental particles, instantons may be dual to space-time events. Event-symmetry is then dual to instanton statistics. In that case a unification between particle statistics and gauge symmetry follows on naturally from the principle of event-symmetry. I build algebras which represent symmetries of superstring theories extending event-symmetry, but which are also isomorphic to an algebra of creation and annihilation operators for strings of fermionic partons.

1. ALGEBRAIC STRING THEORY

Although great strides have been taken toward an understanding of nonperturbative string theory, there is still little progress toward a formulation which shows manifest *general* covariance. In previous work I have tackled the issue by employing the *principle of event-symmetry* as a means of incorporating topology change. Space-time is regarded as a discrete set of events with the permutation group on the events being contained in the universal symmetry of physics. The symmetric group on events trivially contains the diffeomorphism group over any topology (Gibbs, 1996).

It may be that string theory has to be formulated in the absence of space-time, which will then emerge as a derived property of the dynamics. Another interpretation of the event-symmetric approach which embodies this is that instantons are fundamental. Just as solitons may be dual to fundamental particles, instantons may be dual to space-time events or topons (Finkelstein *et al.*, 1997). Event-symmetry is then dual to instanton statistics. In that case

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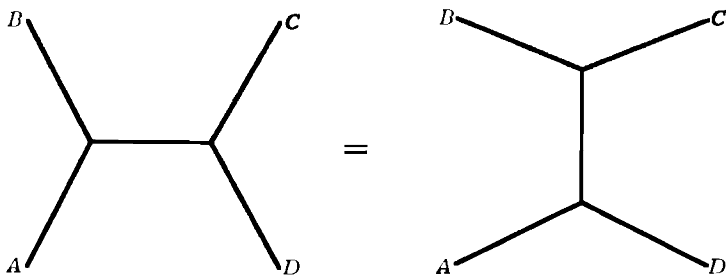
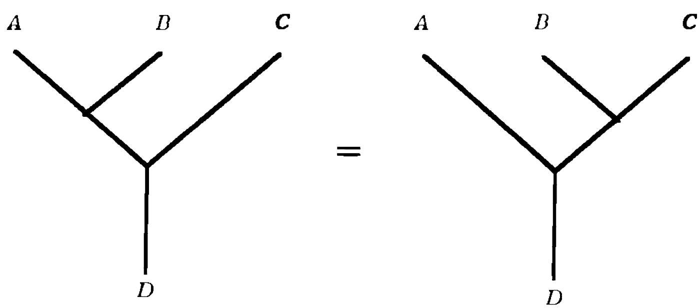
a unification between particle statistics and gauge symmetry follows naturally from the principle of event-symmetry. It is encouraging that this predicted unification now appears in the matrix model of M-theory (Banks *et al.*, 1997).

The final string theory may be founded on a mixture of geometry, topology, and algebra. The dual theory origins of string theory hide a clue to an underlying algebraic nature. In dual theories the s-channel and t-channel amplitudes are supposed to be equal. At tree level, in terms of Feynman diagrams this means that



This diagram could also be distorted to look like



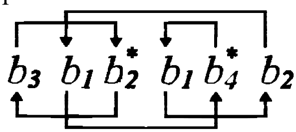
This figure is familiar to many mathematicians, who recognize it as a diagrammatic representation of the associative law,

$$D = (AB)C = A(BC)$$

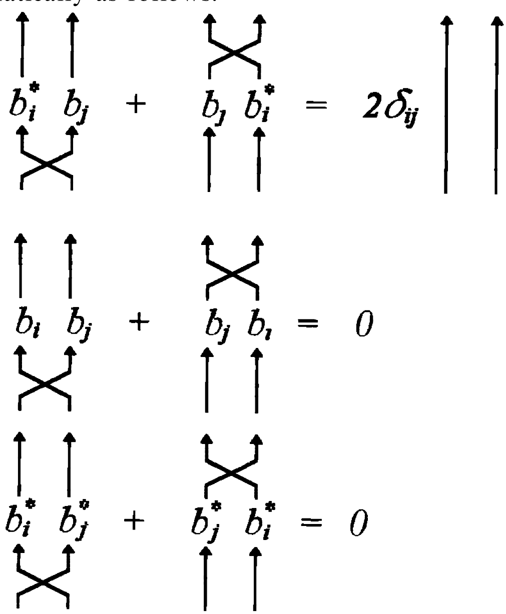
In developing an algebraic string theory the first step would be to define creation and annihilation operators for strings analogous to Dirac's operators for bosonic and fermionic particles. It might be possible to do this if strings are described as composites of particles like a string of beads. The creation and annihilation operators can then be strings of ordinary bosonic or fermionic operators.

2. DISCRETE STRING ALGEBRAS

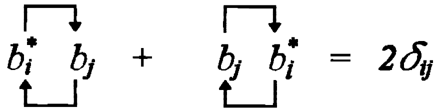
I have previously defined an infinite-dimensional Lie superalgebra for discrete strings which includes event-symmetry (Gibbs, 1996). By an isomorphism which pairs off space-time events and rotates 45 deg in the complex plane of each pair, it is possible to define the desired algebra. The base elements of the algebra consist of an ordered sequence of fermion creation and annihilation operators $b_i\, b_i^*$ linked together by arrows which define an arbitrary permutation. A typical element would look like



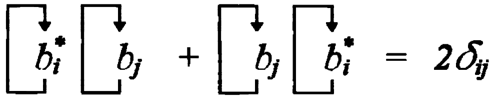
These elements can be multiplied associatively by concatenating them together, but an additional set of relations is enforced which reflect the anticommutation relations of the creation and annihilation operators. They are defined schematically as follows:



Notice that when a creation operator is exchanged with its annihilation partner there is an interaction between the strings. These are partial relations which can be embedded into relations involving complete elements of the algebra. When closed loops which include no operators appear they are identified with unity,

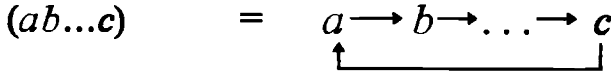


This example shows a cyclic relation on a loop of two operators. The arrows can be joined differently to give another relation,



which is an anticommutation relation for loops of single operators.

By applying these relations repeatedly, it is possible to reorder the operators in any string so that they are reduced to a canonical form of products and sums of ordered cycles. A more convenient notation can be introduced in which an ordered cycle is indicated as follows (where *a, b, c* represent the creation and annihilation operators):



We can generate cyclic relations for loops of any length and graded commutation relations between any pair of strings by repeatedly applying the exchange relations for adjacent pairs of operators. Those relations map the interactions between strings.

The algebra has a Z_2 grading given by the parity of the length of string and it is therefore possible to construct an infinite dimensional Lie-superalgebra using the graded commutator. The algebra may thus be interpreted as both an algebra of creation and annihilation operators and as the supersymmetry algebra of discrete strings.

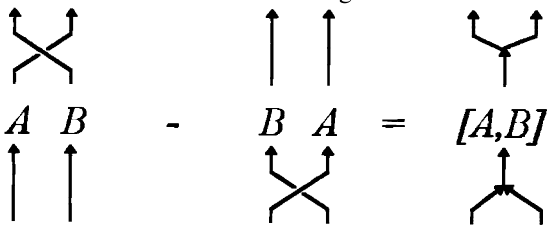
3. SUPERSYMMETRY LADDER

The next stage of the algebraic string theory program is to construct a ladder operation which takes us from one supersymmetry algebra to another one. Starting from the one-dimensional string supersymmetry constructed in the previous section, the ladder operator will take us up to a symmetry of two-dimensional membranes. Further steps take us up to higher dimensional *p*-brane algebras.

We start with a Lie algebra whose elements satisfy the Jacobi relation,

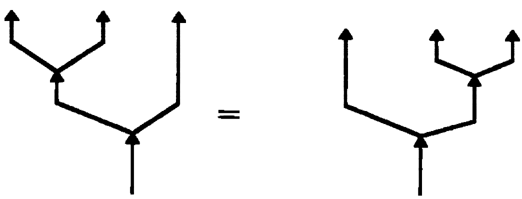
$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$$

A new algebra is constructed by stringing these elements in a sequence and attaching them with an orientated string passing through each one like before. A difference introduced this time is that the string is allowed to have trivalent branches and we must factor out the following relations:

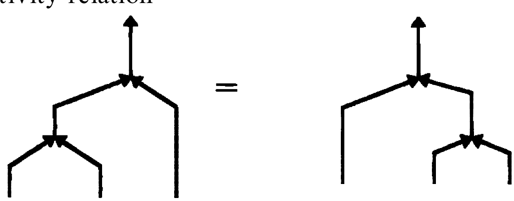


The crossing lines do not (yet) indicate that the lines are knotted. It does not matter which goes over the other.

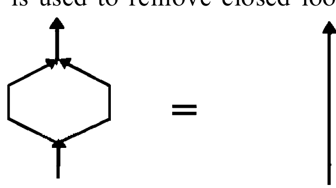
When we check the result of combining the interchanges of three consecutive elements using the commutation relation above, we find that the result is consistent with the Jacobi relation provided we also apply the following associativity relation



and the coassociativity relation



In addition a relationship is used to remove closed loops.



This process defines a new algebra like before except that now we start from any Lie algebra and create a new associative algebra. A new Lie algebra is then defined using the commutators of the algebra as the Lie product.

This construction generalizes easily to Lie superalgebras using graded commutators and graded Jacobi relations. Thus we have a ladder operator which maps one superalgebra to a new one. This ladder operator will be signified by $\boldsymbol{Q}$. Thus if $\mathfrak{A}$ is a Lie superalgebra, then $\boldsymbol{Q}(\mathfrak{A})$ is another.

In the case where we start with the discrete string superalgebra the loops can be visualized as circling the new network. These can then be interpreted as sections of a branching string world sheet. The new algebra is therefore a symmetry for string world sheets or membranes. Application of the ladder operator increases the dimension of the structures each time.

The universal enveloping algebra of the original Lie superalgebra is isomorphic to a subalgebra of the higher dimensional one. That is the subalgebra formed by simply looping each element to itself (the loop-removing relation is needed to establish this). That is, there is a mapping from $\boldsymbol{U}(\mathfrak{A})$ into $\boldsymbol{Q}(\mathfrak{A})$

$$Up: \quad \boldsymbol{U}(\mathfrak{A}) \mapsto \boldsymbol{Q}(\mathfrak{A})$$

Furthermore, there is a homomorphism from the higher algebra to the universal enveloping algebra below, which is defined by removing the string connections:

$$Down: \quad \boldsymbol{Q}(\mathfrak{A}) \to \boldsymbol{U}(\mathfrak{A})$$

It is possible to apply the ladder operator any number n times, giving the algebra $\boldsymbol{Q}^n(\mathfrak{A})$. Since the old algebra is contained in the new, it is also possible to define an algebra $\boldsymbol{Q}^\infty(\mathfrak{A})$ generated by an infinite number of applications of the ladder operator, which then contains all the lower ones. More precisely, $\boldsymbol{Q}\infty(\mathfrak{A})$ is the universal algebra generated from the algebras $\boldsymbol{Q}^n(\mathfrak{A})$, $n = 0, \ldots, \infty$, modulo the identification $X = Up(X)$ for all elements X of the algebra. The algebra $\mathfrak{B} = \boldsymbol{Q}^\infty(\mathfrak{A})$ has the property that applying the ladder operator generates a new one which is isomorphic to the original,

$$\boldsymbol{Q}(\mathfrak{B}) \simeq \mathfrak{B}$$

This raises an interesting question. Starting from a given algebra, is it possible that after only a finite number of applications of the ladder operator, one always arrives at the most complete algebra? Further steps may just create algebras isomorphic to the previous one. This is certainly the case for the algebras $\mathfrak{B}$, which can be generated as above, but in the general case it is an open question.

4. MULTIPLE QUANTIZATION

I would like to propose an interpretation of what the above construction means. The network of connections which appear in the construction of the ladder operator $\boldsymbol{Q}$ could be interpreted as Feynman diagrams. In that case $\boldsymbol{Q}$ can be interpreted as the process of quantization. Here quantization does not just mean a deformation in the sense that a quantum group is a deformation of a Lie group. It means the process of deriving a quantum theory from a classical one as outlined by Dirac using canonical quantization or by Feynman using path integrals.

The application of $\boldsymbol{Q}$ many times is then multiple quantization and suggests a connection with multiple quantization and ur-theory as studied by von Weizsäcker and his collaborators (Görnitz *et al.*, 1992; Lyre, 1996; see also Finkelstein *et al.*, 1997). Ur-theory starts with bits of information, which are quantized to give the group $SU(2)$. This group should be further quantized multiple times to construct a unified theory of physics.

If this interpretation is correct, it also suggests a link between quantization and dimension. The ladder operator $\boldsymbol{Q}$ produces p-brane structures of one higher dimension each time it is applied. It is also quite natural to think of quantization as an operation which generates an extra dimension. Although 4-dimensional classical dynamics is only an approximation to the real 4-dimensional quantum physics, the 3-dimensional kinematic classical state is still preserved in the full quantum theory as the basis of the Hilbert space of states. In quantum field theory we do not usually think of the time dimension as being generated by quantization. It is just the dynamics of the fields in space-time which are generated. However, in quantum gravity, where the structure of space-time is itself part of the dynamics, it *is* natural to regard time as being generated by quantization. Since the spatial dimensions are to be treated the same as the time dimension according to relativity, it is also natural to look to multiple quantization as a mechanism for constructing the dynamics of space-time from more basic foundations.

String theorists have become expert at forming lower dimensional theories from higher dimensional ones by compactification of some of the dimensions. Their difficulty is that they do not have a rigorous foundation for the higher dimensional superstring and p-brane theories they begin with. I suggest that quantization is the operation that can take string theories back up the dimensional ladder. It has been observed that the first-quantized membrane of M-theory in 11 dimensions is the second-quantized string in 10 dimensions (Townsend, 1996). It may be that in general a k-times quantized theory in n dimensions is a $(k + 1)$-times quantized theory in $n - 1$ dimensions. Unlimited multiple quantization may be the way to understanding classical/quantum duality (Duff, 1994).

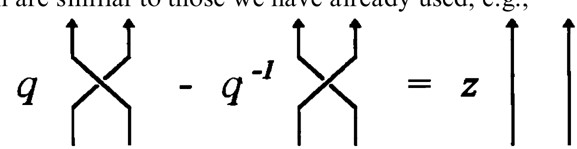
5. FRACTIONAL PARTONS AND KNOTS

The above superstring symmetry construction is all very well, except that strings are not made from discrete fermionic partons. They are defined as continuous loops, but at the same time they may be topological objects which can be determined by discrete points. To try to capture this algebraically it may be necessary to envisage a string as being made from discrete partons with fractional statistics like anyons. Such partons may be repeatedly subdivided into partons with smaller fractional statistics until a continuous limit is found. If strings are truly topological, an infinite sequence of subdivisions may not be necessary. A finite discretization may be sufficient to describe any particular interaction.

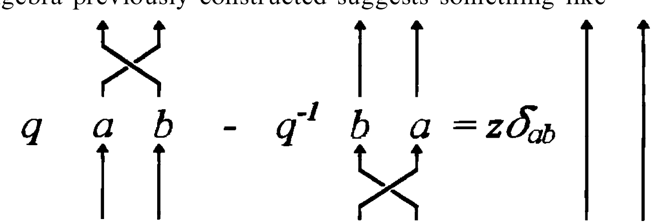
When fractional statistics are introduced the links will have to be replaced by knots, and the supersymmetry algebras will need to be quantized. Now we *are* using the word quantization in the sense of the deformation which is used to construct quantum groups, but this may be related to multiple quantization in a way that is not yet clear.

The use of braided structures will also resolve other problems which are inherent in the use of event-symmetric space-time. If the symmetric group acting on space-time events were part of universal symmetry then it is hard to see how parity could not be conserved, since a mirror reflection of space-time is just a permutation of events. Furthermore, event-symmetry could be used to unravel topological solitons which are so important in string theories, but which depend on the topology of space-time. These difficulties might be resolved if the symmetric group is replaced with the braid group acting on space-time events, especially if these events are tied together with strings which cannot pass through each other.

Motivated by these thoughts, it is natural to seek some kind of deformation of the fermionic string algebra replacing the sign factors in the exchange relations with some general q-parameter. It is also natural to replace the loops which connect the partons with knots. In doing so, we immediately hit upon a fortuitous coincidence. The construction of invariant knot polynomials makes use of Skein relations which are similar to those we have already used, e.g.,



This is the relation which defines the HOMFLY polynomial. Combining this with the algebra previously constructed suggests something like



The special case where there is only one event is related to the HOMFLY polynomial, while the Lie superalgebra previously defined corresponds to the case $q = i$ and $z = 2i$ with the sense in which two strings cross being disregarded. To completely define an algebra, these relations can be embedded in knotted links.

However, the generalisation of the quantization operator $\boldsymbol{Q}$ to strings of braided partons is not straightforward and will be left for future research. To know how to proceed correctly, it is probably necessary to understand the construction in more basic algebraic terms than the combinatorial form in which it is presented above. Further advances may be made using the mathematics of n-category theory. It is thought that n-categories are related to physics in n dimensions (Baez and Dolan, 1995). It is already known that certain 2-categories are applicable to the physics of the string worldsheet. It is natural to conjecture that $(p + 1)$-categories are similarly useful for p-brane world volumes. If the link between multiple quantization and dimension is also correct, then quantization must be defined as a constructor from an n-category to an $(n + 1)$-category which can be applied recursively.

6. CONCLUSIONS

As an initial step toward a purely algebraic formulation of superstring theories I have defined algebras which correspond to both the symmetries and the creation and annihilation operators for strings of discrete fermionic partons. The results suggest a duality between space-time events and instantons as well as a role for multiple quantization in generating space-time dimensions. These are consistent with features of nonperturbative string theories.

It is anticipated that a full algebraic theory will be expressed in the language of n-category theory. The central problem will be to define algebraic quantization as an operator from n-categories to $(n + 1)$-categories. The universal theory may be described by an Ω-category which is isomorphic to its image under quantization.

It is possible that remnants of the symmetries defined here may already lie hidden in the matrix models of M-theory and string theories.

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